(For St Andrews talk, 16/6/10)

Progress on generalized Paparella-Young εpsilon theorems: the strange affair of $\kappa \log \kappa$

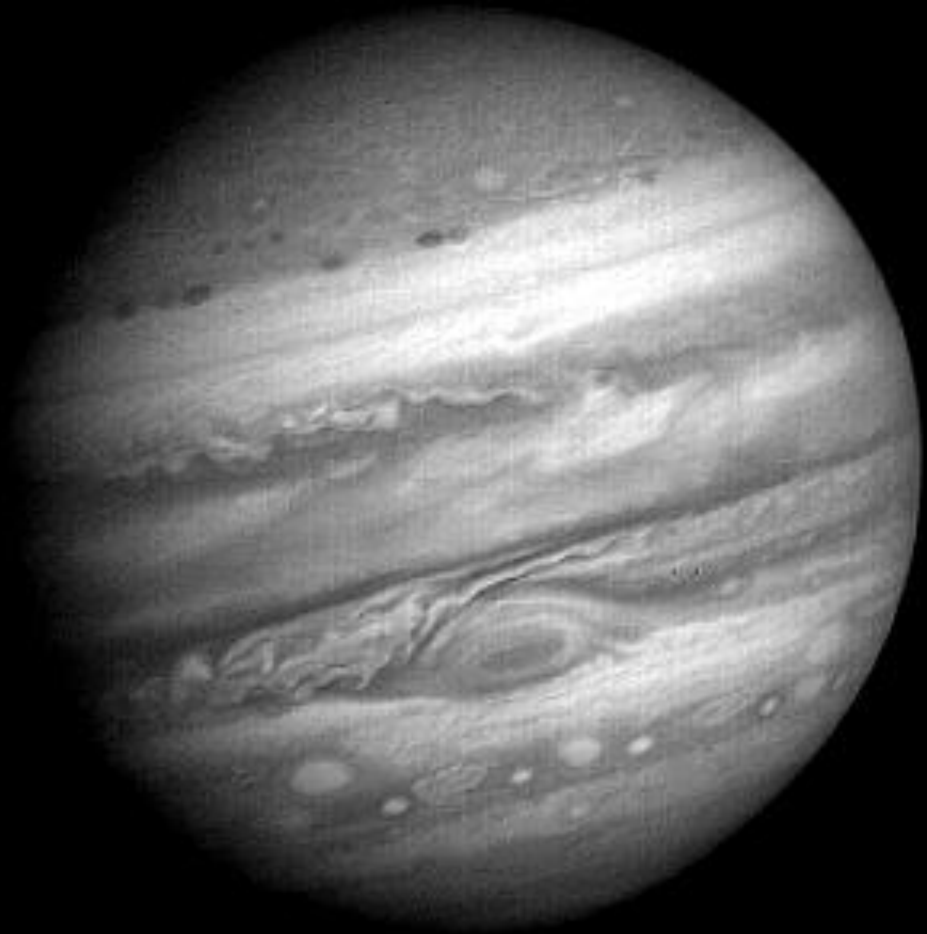
reached slightly circuitously via echoes of the **ozone hole** and remarks on the **amplifier principle**,

some of the εpsilon stuff having first been reported at the IUTAM/Newton-Inst. Workshop on
Rotating Stratified Turbulence and Turbulence in the Atmosphere and Oceans,
December 2008, Isaac Newton Institute, Cambridge
and independently discovered by Jonas Nycander
www.atm.damtp.cam.ac.uk/people/mem#mixing

Michael E. McIntyre

DAMTP and Centre for Atmospheric Science,
University of Cambridge

but first, even more circuitously, a quick update on 2 other topics…



www.atm.damtp.cam.ac.uk/people/mem#jupiter1

Voyager 1 approaching (60 Jupiter days)

Richard Wood and I consider "generalized mixing" of q-g PV
(e.g. Pasquill & Smith 1983, Fiedler 1984, Stull 1984, Shnirelman 1993,
Esler, Willcocks in prep)

"later" "initial" q-g PV

$$q_\ell(x,y) \quad = \quad \iint_{\mathcal{D}} dx'dy'\, q_{\rm i}(x',y')\, R(x',y';x,y)$$

"mixing kernel"

$$\iint_{\mathcal{D}} dx\,dy\, R(x',y';x,y) \quad = \quad 1 \quad \forall\, (x',y') \in \mathcal{D}$$

$$R(x',y';x,y) \quad \geq \quad 0 \quad \forall\, (x,y), (x',y') \in \mathcal{D}$$

$$\iint_{\mathcal{D}} dx'dy'\, R(x',y';x,y) \quad = \quad 1 \quad \forall\, (x,y) \in \mathcal{D}\ .$$

–``bistochastic"

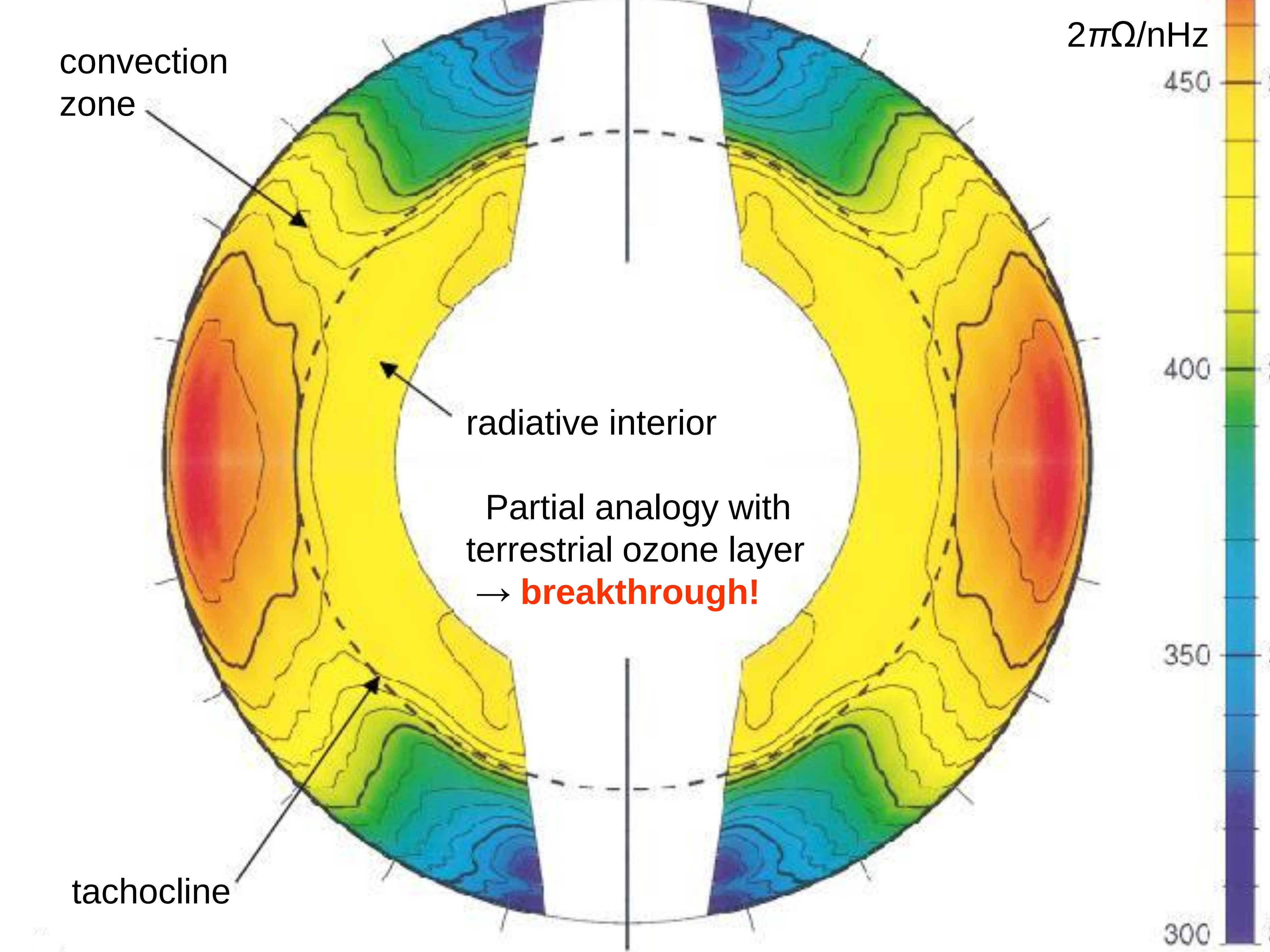
Theorem: within quasigeostrophic dynamics,
ANY generalized PV mixing event starting from a zonally symmetric initial state with $dq_{\rm i}/dy > 0$ **causes a –ve angular momentum change**

(Wood R. B. & McI 2010: `A general theorem…', J. Atmos.Sci. **67**, 1261
with APE and **CORRECTED** Arnol'd-theorem spinoffs.

www.atm.damtp.cam.ac.uk/people/mem#jupiter1

PV mixing etc is relevant to this system too, albeit indirectly…

NASA/SOHO


2πΩ/nHz
450
400
350
300
convection zone
radiative interior
tachocline
Partial analogy with terrestrial ozone layer
→ **breakthrough!**

Coming back to Earth: something absurdly simple:

This simple thing is the **audio-amplifier principle** (unknown to climate skeptics). Or maybe it should just be called the **amplifier principle**:

Small inputs can have large effects (!)

(Note the perils of energy arguments.)

Key distinction: "**input signal**" versus "**internal variable**"

Not an absolute distinction of course – as always, it depends on what experiment, or thought-experiment, you're doing.

Three examples:

Input signals on the left; **internal variables** in the middle::

Inject electrons

Audio amplifier:
Large internal flows of holes and electrons

Inject CFC molecules

Ozone layer:
Large internal inter-species flow from O_3 to O_2 (catalytic)

Inject CO_2 molecules

orbital forcing

Climate system:
Large internal flows of, e.g., water vapour and active carbon

Large output signals

(This input **not** small by any natural measure: (180 $\leftrightarrow$ 280 $\rightarrow$ 380+ ppmv)

(watch this space – work with Dr Luke Skinner)

Input signals on the left; **internal variables** in the middle:

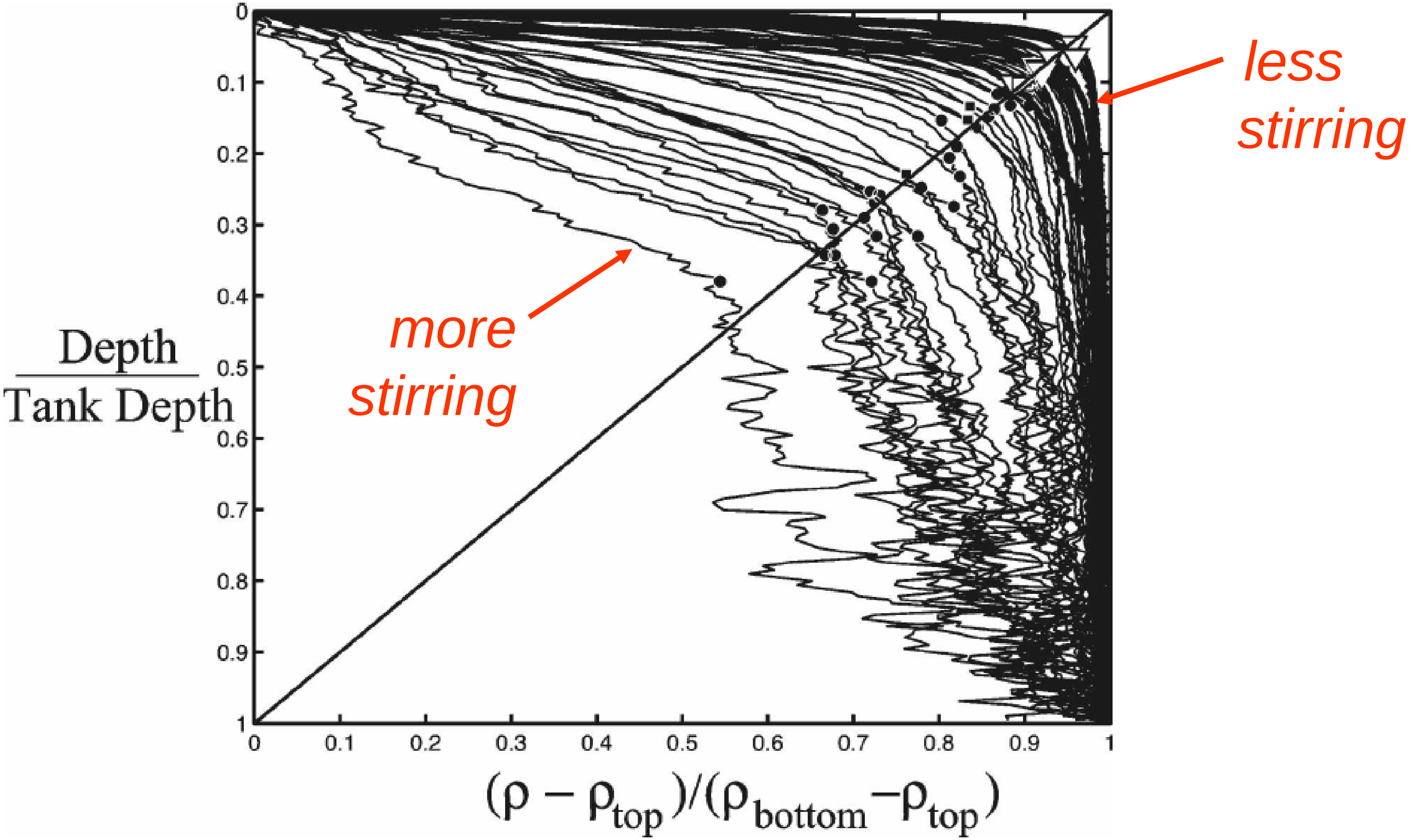
Inject few TW of mechanical stirring → **Oceans: Large internal flows of heat & active carbon** → **Large climate change**

The way this particular input signal works is well illustrated by a recent lab experiment on "**horizontal convection**" plus stirring.

Buoyancy forcing was applied at the top surface of a tank while the whole tank was mechanically stirred, at different rates.

(NB: zero mechanical stirring doesn't imply zero motion – just weaker motion and stratification.)

Lab expt: Whitehead and Wang 2008 (JPO **38**,1091):

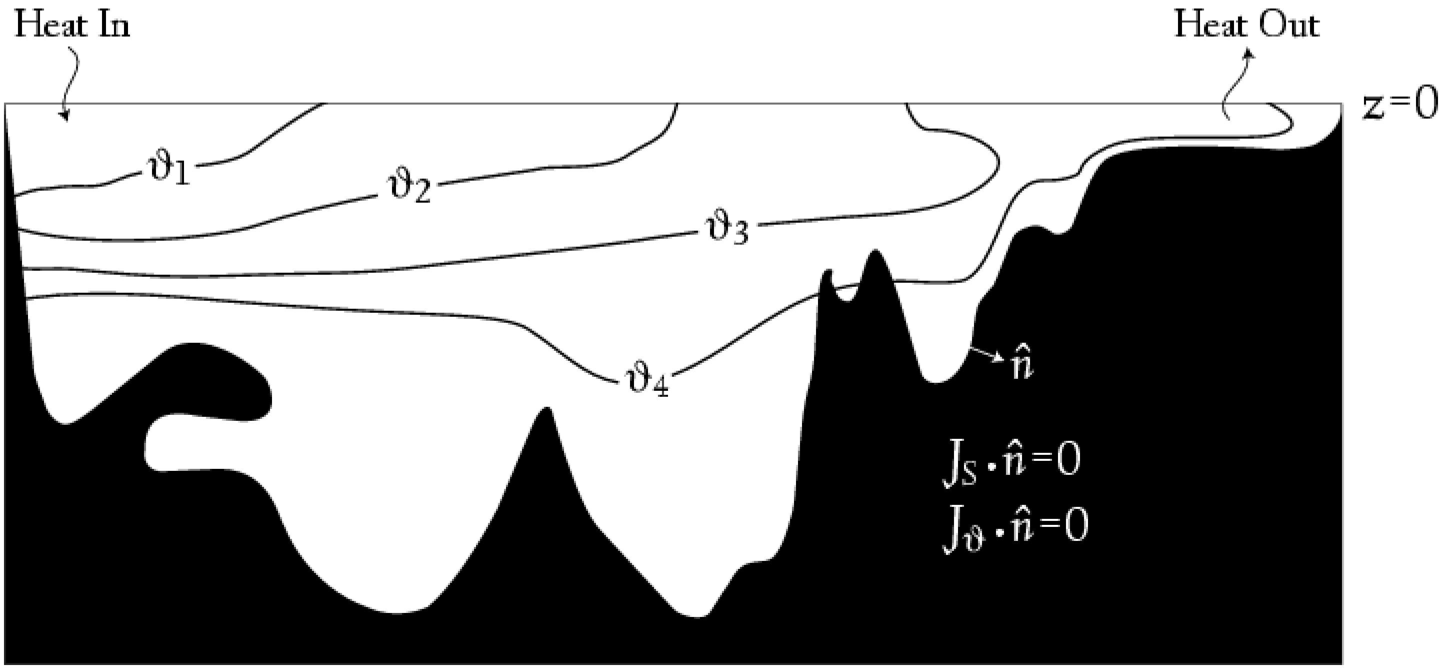


Now think about
realistic oceans minus mechanical stirring:

***T* and *S* distributions are prescribed, or relaxed toward,**

at top surface, so convective stirring only:

warmer &/or fresher **colder &/or saltier**

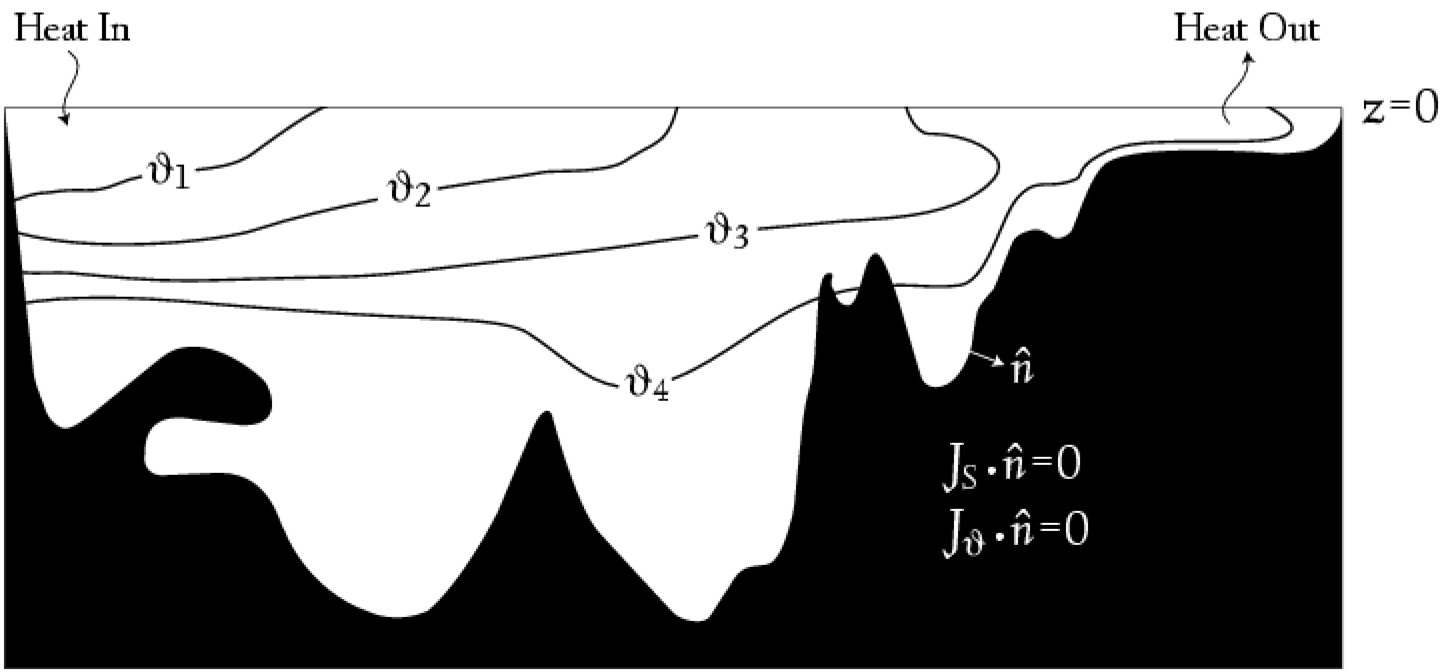


A key contribution was Munk & Wunsch 1998 (*Deep-Sea Res.* **45**, 1977) – avoided getting bogged down in energetics and losing sight of the **amplifier principle**.

T **and** ***S*** **distributions are prescribed, or relaxed toward,**

at top surface, so convective stirring only:

warmer &/or fresher **colder &/or saltier**



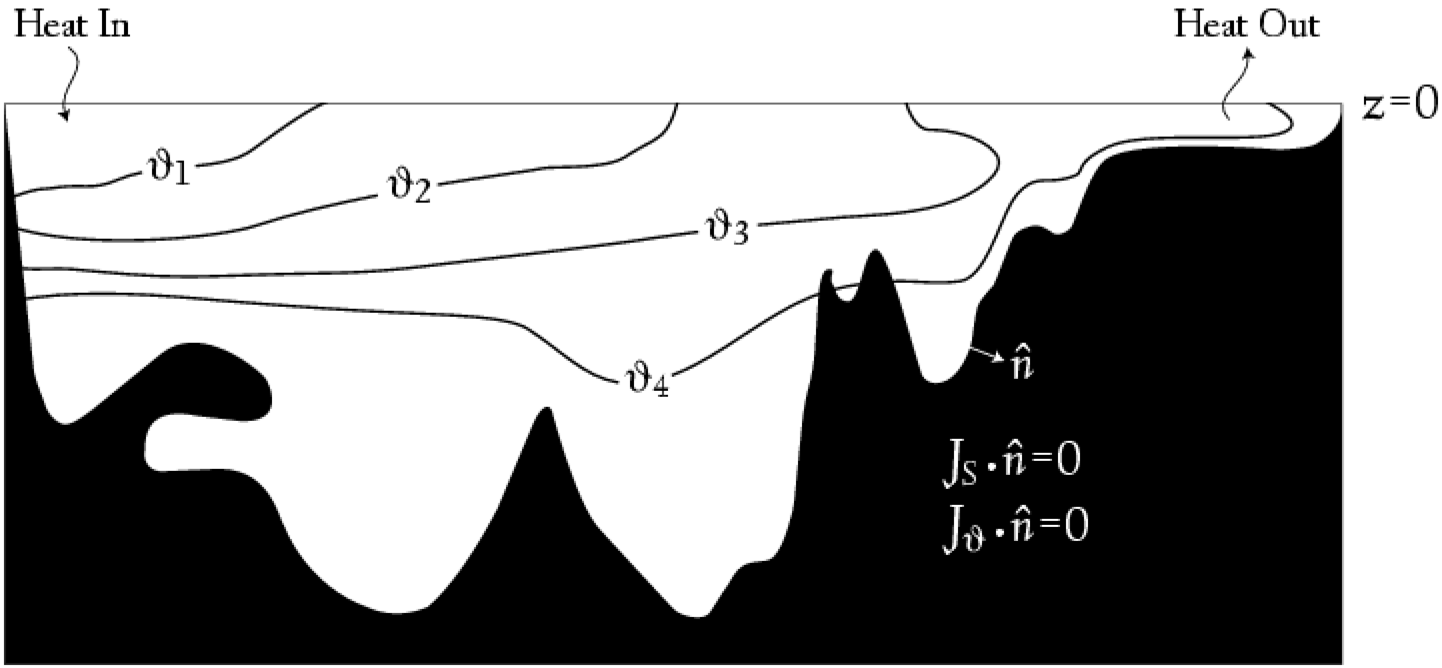
Munk & Wunsch 1998 (*DSR* **45**, 1977) argue heuristically that, with no mechanical stirring, the bulk of the ocean would become **unstratified, at maximal "density"** (& weaker transporter of heat, nutrients, CO_2 etc) The (remarkable!) **Paparella-Young theorem** promises to put some **useful constraints on "weaker"**, in terms of *ε*.

However, the original PY 2002 analysis is for a very idealized case only:

T **and** ***S*** **distributions are prescribed, or relaxed toward,**

at top surface, so convective stirring only:

warmer &/or fresher **colder &/or saltier**



Theorem (Paparella and Young 2002, *J. Fluid Mech*. **466**, 205):
In **a purely thermal version of this thought-experiment with a linear EOS in a rectangular box**, $\langle\langle \varepsilon \rangle\rangle$ goes to zero **like the thermal diffusivity** – in the "standard limit" of small diffusivity κ holding Prandtl number constant.

as I'll call it

2-dimensional numerical experiments give some idea of what happens in this idealized case (purely thermal forcing, zero mechanical stirring). Motion is nontrivial (Francesco Paparella, personal communication):

$$Ra = 10^8\,, \quad Pr = \nu/\kappa = 10$$

PY02's result:

Volume and time averaged turbulent dissipation rate

thermal diffusivity

$$\langle\langle \varepsilon \rangle\rangle \leqslant \kappa \Delta b / H$$

depth of box

NOTE: epsilon goes to zero like the FIRST power of κ

This is related to abyssal mixing & MOC rates through the
Ellison-Britter-Osborn empirical mixing formula

$$K_z \lesssim \gamma \varepsilon / N^2 \qquad \gamma \sim 0.2$$

(as distinct from Osborn-Cox relation,

$$K_z = \kappa \, |\nabla \vartheta|^2 / \bar{\vartheta}_{,z}^{\,2}$$

As well as idealized geometry & uniform gravity, PY02 assumed:

- a linear EOS with a single buoyancy agent, *T* only or *S* only (no cabbeling, no double diffusion, no thermobarics)
- (Oberbeck-)Boussinesq dynamics (infinite sound speed, inertial density constant)

All these restrictions have now been lifted (ongoing work in collaboration with Francesco Paparella and William R. Young, www.atm.damtp.cam.ac.uk/people/mem/#mixing and independent work by Jonas Nycander). Main keys to progress were:

- Dissect the complete energetics in a certain way, using the notion of dynamic enthalpy. (W. R. Young 2010, *J. Phys. Oc*. **40**, 394-400).
- Use: McDougall's conservative temperature (aka potential enthalpy) in place of temperature or potential temperature. The full thermodynamics of diffusion can then be used in the most "simple" yet accurate way.

(Boussinesq case)

scaled geopotential

$$
\begin{aligned}
\mathrm{D}\mathbf{u}/\mathrm{D}t \;+\; 2\boldsymbol{\Omega}\times\mathbf{u} \;+\; \boldsymbol{\nabla}p \;-\; b\boldsymbol{\nabla}\mathcal{Z} &= \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} \\
\mathrm{D}\vartheta/\mathrm{D}t &= -\boldsymbol{\nabla}\cdot\mathbf{J}_{\vartheta} \\
\mathrm{D}S/\mathrm{D}t &= -\boldsymbol{\nabla}\cdot\mathbf{J}_{S} \\
\boldsymbol{\nabla}\cdot\mathbf{u} &= 0 \\
b &= b(\vartheta, \mathcal{Z}, S)
\end{aligned}
$$

$$
\Rightarrow\; \frac{\partial}{\partial t}\left(\tfrac{1}{2}|\mathbf{u}|^2\right) \;-\; \boxed{wb} \;+\; \boldsymbol{\nabla}\cdot\left\{\mathbf{u}\left(\tfrac{1}{2}|\mathbf{u}|^2 + p\right) - \mathbf{u}\cdot\boldsymbol{\sigma}\right\} \;=\; -\varepsilon
$$

$$
w := \mathrm{D}\mathcal{Z}/\mathrm{D}t \qquad\qquad \varepsilon := \boldsymbol{\nabla}\mathbf{u}:\boldsymbol{\sigma} := u_{i,j}\sigma_{ij}
$$

What now?

*(This is **not** D/Dt of $b\mathcal{Z}$)*

Young's "dynamic enthalpy":

JPO **40**, 394 (2010)

= 0 at top surface

$$h^{\ddagger}(\vartheta, \mathcal{Z}, S) \;:=\; \int_{\mathcal{Z}}^{0} b(\vartheta, \mathcal{Z}', S)\, \mathrm{d}\mathcal{Z}'$$

$$\Rightarrow \qquad \frac{\mathrm{D}h^{\ddagger}}{\mathrm{D}t} \;=\; -\boxed{wb} + \mathcal{D}(\vartheta, \mathcal{Z}, S)$$

where

$$\mathcal{D}(\vartheta, \mathcal{Z}, S) \;:=\; \frac{\partial h^{\ddagger}}{\partial \vartheta}\frac{\mathrm{D}\vartheta}{\mathrm{D}t} \;+\; \frac{\partial h^{\ddagger}}{\partial S}\frac{\mathrm{D}S}{\mathrm{D}t}$$

→ 0 at top surface, like $\mathcal{Z}$

$$= \; -\frac{\partial h^{\ddagger}}{\partial \vartheta}\,\boldsymbol{\nabla}\cdot\mathbf{J}_{\vartheta} - \frac{\partial h^{\ddagger}}{\partial S}\,\boldsymbol{\nabla}\cdot\mathbf{J}_{S}$$

$$\Rightarrow$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\mathbf{u}|^2 + h^\ddagger\right) + \boldsymbol{\nabla}\cdot\left\{\mathbf{u}\left(\tfrac{1}{2}|\mathbf{u}|^2 + p + h^\ddagger\right) - \mathbf{u}\cdot\boldsymbol{\sigma}\right\} = -\varepsilon + $$

$$+\,\mathcal{D}(\vartheta,\, \mathcal{Z},\, S)$$

$$\Rightarrow \qquad \langle\!\langle \varepsilon \rangle\!\rangle = \langle\!\langle \mathcal{D}(\vartheta,\, \mathcal{Z},\, S) \rangle\!\rangle$$

using b.c.'s & stat. steadiness

= 0 at top surface

$$\Rightarrow \quad \langle\!\langle \varepsilon \rangle\!\rangle = \langle\!\langle\, \mathbf{J}_\vartheta\cdot\boldsymbol{\nabla}(\partial h^\ddagger/\partial\vartheta) + \mathbf{J}_S\cdot\boldsymbol{\nabla}(\partial h^\ddagger/\partial S)\,\rangle\!\rangle$$

(normal cpts = 0 on topography)

$$\Rightarrow$$

$$\langle\!\langle \varepsilon \rangle\!\rangle + \boxed{\kappa_0\,\langle\!\langle\, A|\boldsymbol{\nabla}\vartheta|^2 + B\boldsymbol{\nabla}\vartheta\cdot\boldsymbol{\nabla}S + C|\boldsymbol{\nabla}S|^2 \rangle\!\rangle}$$

$$= -\kappa_0\langle\!\langle D\boldsymbol{\nabla}\vartheta\cdot\boldsymbol{\nabla}\mathcal{Z} + E\boldsymbol{\nabla}S\cdot\boldsymbol{\nabla}\mathcal{Z} + F|\boldsymbol{\nabla}\mathcal{Z}|^2 \rangle\!\rangle$$

Can bound 2nd line but not 1st – **squared gradients can be catastrophically large!**

So epsilon theorems **not** provable unless

$$A \geqslant 0\,, \qquad C \geqslant 0\,, \qquad \text{and} \qquad B^2 - 4AC \leqslant 0$$

$$\langle\langle \varepsilon \rangle\rangle + \kappa_0 \langle\langle A|\nabla\vartheta|^2 + B\nabla\vartheta\cdot\nabla S + C|\nabla S|^2 \rangle\rangle$$
$$= -\kappa_0 \langle\langle \boxed{D\nabla\vartheta\cdot\nabla z} + E\nabla S\cdot\nabla z + F|\nabla z|^2 \rangle\rangle$$

For at least some highly realistic ocean models, ***TRUE*** *that*

$$A \geqslant 0\,, \quad C \geqslant 0\,, \quad \text{and} \quad B^2 - 4AC \leqslant 0 \quad \textbf{(!!)}$$

Then, if 2nd line boundable, we can constrain not only $\langle\langle \varepsilon \rangle\rangle$ **but also**

$$|\nabla\vartheta|^2 \quad \text{and} \quad |\nabla S|^2 \qquad \textbf{(!)}$$

BUT! not uniformly: A, B, C go like $|z|$ near the top surface. (They inherit this property from Young's dynamic enthalpy.)

($\text{PY02's case:} \quad A = B = C = E = F = 0\,, \quad D = \text{constant}$

and ∇z is a unit vertical vector, so the D term integrates to an order-unity quantity in the standard limit, $\kappa_0 \to 0$ holding other parameters constant.)

A specific example (details in my IUTAM/Newton conference paper, www.atm.damtp.cam.ac.uk/people/mem/#mixing):

$$\mathbf{J}_\vartheta = -\kappa\,(\boldsymbol{\nabla}\vartheta - \boxed{\Gamma_\vartheta \boldsymbol{\nabla} z})\ , \qquad \mathbf{J}_S = -\kappa_S\,(\boldsymbol{\nabla} S + \boxed{\Gamma_S \boldsymbol{\nabla} z})\ ,$$

$$\Gamma_\vartheta \sim 0.15\,\mathrm{K\,km^{-1}}\ , \qquad \Gamma_S \sim -3\,‰\,\mathrm{km^{-1}}$$

$$b(\vartheta, z, S) = g_0 \left\{ \beta_\vartheta (1 - \gamma^* \rho_0 g_0 \boxed{z)\vartheta} + \boxed{\tfrac{1}{2}\beta_\vartheta^* \vartheta^2} - \beta_S S + \boxed{\tfrac{1}{2}\beta_S^* S^2} + \frac{g_0 z}{c_0^2} \right\}$$

$$\Rightarrow\ h^\ddagger = -g_0 z \left\{ \beta_\vartheta \left(1 - \tfrac{1}{2}\gamma^* \rho_0 g_0 z\right) \vartheta + \tfrac{1}{2}\beta_\vartheta^* \vartheta^2 - \beta_S S + \tfrac{1}{2}\beta_S^* S^2 + \frac{g_0 z}{2c_0^2} \right\}$$

$$\frac{\partial h^\ddagger}{\partial \vartheta} = -g_0 z\left(\beta_\vartheta - \tfrac{1}{2}\gamma^* \beta_\vartheta \rho_0 g_0 z + \beta_\vartheta^* \vartheta\right)\ , \qquad \frac{\partial h^\ddagger}{\partial S} = g_0 z(\beta_S - \beta_S^* S)$$

$$\Rightarrow\ \boldsymbol{\nabla}\frac{\partial h^\ddagger}{\partial \vartheta} = -g_0 z \beta_\vartheta^* \boxed{\boldsymbol{\nabla}\vartheta} - g_0\left(\beta_\vartheta - \gamma^* \beta_\vartheta \rho_0 g_0 z + \beta_\vartheta^* \vartheta\right) \boldsymbol{\nabla} z\ ,$$

$$\boldsymbol{\nabla}\frac{\partial h^\ddagger}{\partial S} = -g_0 z \beta_S^* \boxed{\boldsymbol{\nabla} S} + g_0\left(\beta_S - \beta_S^* S\right) \boldsymbol{\nabla} z$$

$$\langle\!\langle \varepsilon \rangle\!\rangle + \kappa_0 \langle\!\langle A|\nabla\vartheta|^2 + B\nabla\vartheta\cdot\nabla S + C|\nabla S|^2 \rangle\!\rangle$$
$$= -\langle\!\langle D\nabla\vartheta\cdot\nabla z + E\nabla S\cdot\nabla z + F|\nabla z|^2 \rangle\!\rangle$$

where:

$$A = \hat\kappa_\vartheta \beta^*_\vartheta g_0 |z| \ ,$$

$$B = 0 \ ,$$

$$C = \hat\kappa_S \beta^*_S g_0 |z| \ ,$$

(both positive!)

(& again note behaviour at top surface)

$$D = -\hat\kappa_\vartheta g_0 \big(\beta_\vartheta + \gamma^* \beta_\vartheta \rho_0 g_0 |z| + \beta^*_\vartheta \vartheta\big) - \hat\kappa_\vartheta \boxed{\Gamma_\vartheta} \beta^*_\vartheta g_0 |z| \ ,$$

$$E = \hat\kappa_S g_0 \left(\beta_S - \beta^*_S S\right) + \hat\kappa_S \boxed{\Gamma_S} \beta^*_S g_0 |z| \ ,$$

$$F = \hat\kappa_\vartheta g_0 \boxed{\Gamma_\vartheta} \big(\beta_\vartheta + \gamma^* \beta_\vartheta \rho_0 g_0 |z| + \beta^*_\vartheta \vartheta\big) + \hat\kappa_S g_0 \boxed{\Gamma_S} \big(\beta_S - \beta^*_S S\big)$$

The standard limit $\kappa_0 \to 0$ holds other quantities constant **including**

$$\hat\kappa_\vartheta := \kappa/\kappa_0 \ , \qquad \hat\kappa_S := \kappa_S/\kappa_0 \qquad \text{(order-unity quantities)}$$

In this case we can perform the integrations on the 2nd line above, and **throw away the squared-gradient information** on the 1st line to get

$$\boxed{\langle\!\langle \varepsilon \rangle\!\rangle = O(\kappa_0)}$$

(cf. Nycander)

But: this proof depends on domain-integrability of, e.g., $\vartheta\boldsymbol{\nabla}\vartheta$.
So it fails for any fully realistic EOS with variable coefficients.

However, with slightly stronger assumptions about $4AC - B^2$ **and** about conditions near the surface, we can prove that in the standard limit

$$\boxed{\langle\!\langle \varepsilon \rangle\!\rangle \;=\; O(\kappa_0 |\ln \kappa_0|)\ .}$$

Recall that $\langle\!\langle \varepsilon \rangle\!\rangle + \kappa_0 Q + \kappa_0 R = 0$,

where
$$Q \;:=\; Q(\kappa_0) \;=\; \langle\!\langle A|\boldsymbol{\nabla}\vartheta|^2 \;+\; B\boldsymbol{\nabla}\vartheta\boldsymbol{\cdot}\boldsymbol{\nabla}S \;+\; C|\boldsymbol{\nabla}S|^2\rangle\!\rangle\ ,$$
$$R \;:=\; R(\kappa_0) \;=\; \langle\!\langle D\boldsymbol{\nabla}\vartheta\boldsymbol{\cdot}\boldsymbol{\nabla}z \;+\; E\boldsymbol{\nabla}S\boldsymbol{\cdot}\boldsymbol{\nabla}z \;+\; F|\boldsymbol{\nabla}z|^2\rangle\!\rangle$$

Assume:

$$\inf(A/|z|) > 0\ ,\quad \inf(C/|z|) > 0\ ,\quad \inf\left\{(4AC - B^2)/|z|\right\} > 0$$

and: $(\Rightarrow\ 0 \leqslant Q \leqslant |R|)$

Skin-depth assumption: There is a top layer $0 \geqslant z \geqslant -\Delta z$ that contributes only $O(1)$ to Q and R as $\kappa_0 \to 0$, with the (positive) layer thickness Δz not too small, specifically

$$\Delta z \;\geqslant\; \text{const.} \times \kappa_0^p \tag{20}$$

for some positive power p (which can be 0.3, 1, 10, 1000, or any fixed positive #).

Denoting the volume of the domain by V, we slice the domain integrals into layer contributions and use the skin-depth assumption:

$$Q = \frac{1}{V}\int_{\mathcal{Z}_b}^{-\Delta\mathcal{Z}}\iint\langle A|\nabla\vartheta|^2 + B\nabla\vartheta\cdot\nabla S + C|\nabla S|^2\rangle\, d\mathcal{A}\, d\mathcal{Z} + O(1)$$

$$R = \frac{1}{V}\int_{\mathcal{Z}_b}^{-\Delta\mathcal{Z}}\iint\langle D\nabla\vartheta\cdot\nabla\mathcal{Z} + E\nabla S\cdot\nabla\mathcal{Z} + F|\nabla\mathcal{Z}|^2\rangle\, d\mathcal{A}\, d\mathcal{Z} + O(1)$$

Simplest nontrivial case is $B = C = E = 0$. Use **layerwise Cauchy-Schwarz:**

$$\frac{1}{V}\left|\iint\langle D\nabla\vartheta\cdot\nabla\mathcal{Z}\rangle\, d\mathcal{A}\right| \leqslant D'\boxed{\left(\iint\langle|\nabla\vartheta|^2\rangle\, d\mathcal{A}\right)^{1/2}} = \Lambda(\mathcal{Z}, \kappa_0) \text{ say}$$

where $D' = V^{-1}\sup_{\mathcal{Z}}\left(\iint\langle|D\nabla\mathcal{Z}|^2\rangle\, d\mathcal{A}\right)^{1/2}$, an order-unity constant.

Thus

$$|R| \leqslant D'\int_{\mathcal{Z}_b}^{-\Delta\mathcal{Z}}\Lambda(\mathcal{Z}, \kappa_0)\, d\mathcal{Z} + O(1)$$

so

$$\left.\begin{aligned}\inf(A/|Z|) > 0\\ Q \leqslant |R|\end{aligned}\right\} \Rightarrow \int_{\mathcal{Z}_b}^{-\Delta\mathcal{Z}}|\mathcal{Z}|\,\Lambda^2\, d\mathcal{Z} \leqslant D''\int_{\mathcal{Z}_b}^{-\Delta\mathcal{Z}}\Lambda\, d\mathcal{Z} + O(1)$$

another order-unity constant

$$\int_{z_{\rm b}}^{-\Delta z} |z|\,\Lambda^2 dz \;\leqslant\; D'' \int_{z_{\rm b}}^{-\Delta z} \Lambda\, dz \;+\; O(1)$$

If RHS bounded in the standard limit, then nothing to prove. If unbounded, then it can get large only like $|\ln \kappa_0|$. To prove this, define

$$\lambda \;:=\; |z|\,\Lambda\;,$$

$$\ell \;:=\; -\ln|z|\;,$$

and

$$\ell_{\rm b} \;:=\; -\ln|z_{\rm b}|\;, \qquad L \;:=\; -\ln|\Delta z|\;,$$

noting that $\ell_{\rm b} = O(1)$ but $L \to \infty$ in the limit. Then

$$\int_{\ell_{\rm b}}^{L} \lambda^2 d\ell \;\leqslant\; D'' \int_{\ell_{\rm b}}^{L} \lambda\, d\ell \;+\; O(1)\;.$$

Divide by $(L - \ell_{\rm b})$;
use (square of mean) < (mean of square):

$$\big(\overline{\lambda}\big)^2 \;\leqslant\; D''\,\overline{\lambda} \;+\; O(L^{-1})\;,$$

which with L large implies that $\overline{\lambda} = O(1)$ and therefore that

$$\int_{\ell_{\rm b}}^{L} \lambda d\ell \;=\; O(L) \quad \text{as } \kappa_0 \to 0\;,$$

i.e. like $|\ln \kappa_0|$ as asserted. So finally

$$\boxed{\langle\langle \varepsilon \rangle\rangle \;=\; O(\kappa_0 |\ln \kappa_0|)\ .}$$

Next steps:

- Extend to fully compressible (non-Boussinesq) equations; need a further assumption, that P stratifies the ocean and, e.g., no inverse barometric stirring

- Extend to SCOR/IAPSO WG 127 equation of state and thermodynamics **– definiteness assumption hangs by a thread!!**

- Numerical rather than asymptotic estimates, including finite depth of penetration of solar heating (few metres on average).